Modal Testing and Analysis

Undamped MDOF systems

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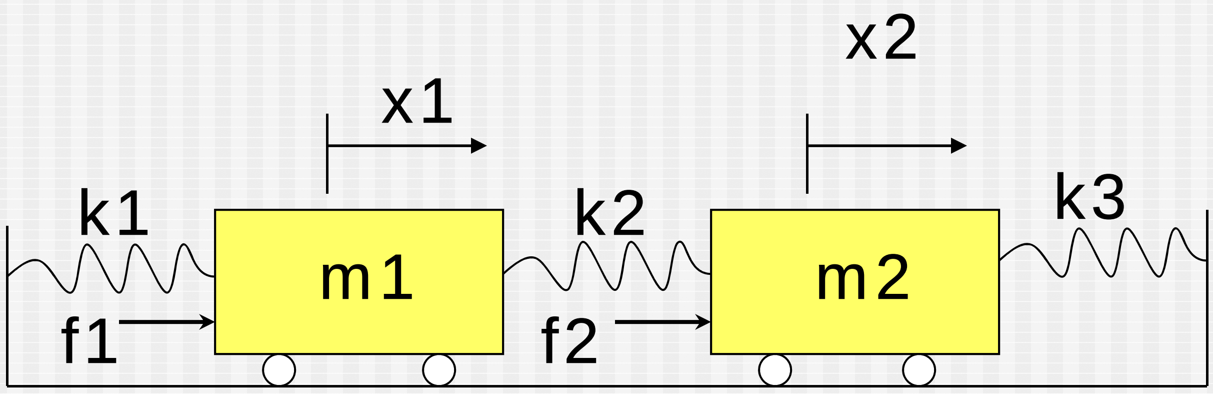
Undamped MDOF systems

$$[M]\{\ddot{x}\}+[K]\{x\}=\{f(t)\}$$

[M] and [K] are N*N mass and stiffness matrices.

{f(t)} is N*1 force vector



Undamped 2DOF system

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = f_1$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = f_2$$

or

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$$m_1 = m_2 = 1kg$$

$$k_1 = k_3 = 0.4MN/m$$

$$k_2 = 0.8MN/m$$

Undamped 2DOF system

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 1200000 & -800000 \\ -800000 & 800000 \end{bmatrix}$$

MDOF- Free Vibration

$$[M]\{\ddot{x}\}+[K]\{x\}=\{f(t)\}$$

$$\{f(t)\}=0$$

$$\{x(t)\}=\{X\}e^{i\omega t}$$

$$([K]-\omega^2[M])\{X\}e^{i\omega t}=\{0\}$$

$$To\ have\ non-trvial\ solution:$$

$$\det|[K]-\omega^2[M]|=0$$

MDOF- Free Vibration

Natural Frequencies:

$$\omega_1^2, \omega_2^2, \ldots, \omega_r^2, \ldots, \omega_N^2$$

Mode shapes:

$$\psi_1, \psi_2, \ldots, \psi_r, \ldots, \psi_N$$

Or in matrix form:

$$\left[\ddots \omega_r^2 \ddots\right], \left[\Psi\right]$$ Modal Model

2DOF System- Free Vibration

$$\omega^4 - (2.4\times10^6)\omega^2 + (0.8\times10^{12}) = 0$$

Solving the equation:

$$\omega_1^2 = 4\times10^5 \quad (Rad/s)^2$$

$$\omega_2^2 = 2\times10^6 \quad (Rad/s)^2$$

Numerically:

$$\left[\omega_r^2\right] = \begin{bmatrix} 4\times10^5 & 0 \\ 0 & 2\times10^6 \end{bmatrix}$$

$$[\Psi] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Orthogonality Properties of MDOF

The modal model possesses some very important Properties, stated as:

$$[\Psi]^T [M][\Psi] = \left[\ddots\, m_r \ddots\right] \quad \text{Modal mass matrix}$$

$$[\Psi]^T [K][\Psi] = \left[\ddots\, k_r \ddots\right] \quad \text{Modal stiffness matrix}$$

$$\left[\omega_r^2\right] = [m_r]^{-1}[k_r]$$

Exercise: Prove the orthogonality property of MDOF

Mass-normalisation

The mass normalized eigenvectores are written as $[\Phi]$
And have the following property:

$$[\Phi]^T[M][\Phi] = \begin{bmatrix} \ddots & & \\ & I & \\ & & \ddots \end{bmatrix}$$

$$[\Phi]^T[K][\Phi] = \begin{bmatrix} \ddots & & \\ & \omega_r^2 & \\ & & \ddots \end{bmatrix}$$

The relationship between mass normalised mode shape and its more general form $[\Psi]$ is:

$$\{\phi_r\} = \frac{1}{\sqrt{m_r}}\{\psi_r\} \quad where \quad m_r = \{\psi_r\}^T[M]\{\psi_r\}$$

$$[\Phi] = [\Psi]\left[m_r^{-1/2}\right]$$

Mass-normalisation of 2DOF

$$[\Psi]^T[M][\Psi]=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=[m_r]$$

$$[\Psi]^T[K][\Psi]=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1.2 & -.8\\ -.8 & 1.2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}10^6=\begin{bmatrix}.8 & 0\\ 0 & 4\end{bmatrix}10^6=[k_r]$$

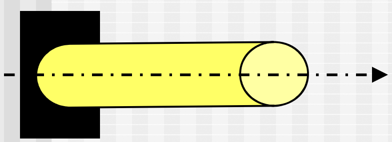
Clearly:

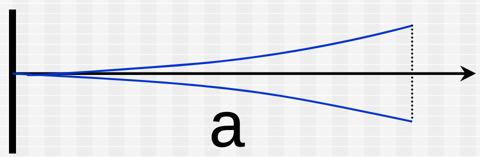
$$[m_r]^{-1}[k_r]=\begin{bmatrix}0.4 & 0\\ 0 & 2\end{bmatrix}10^6=\left[\omega_r^2\right]$$

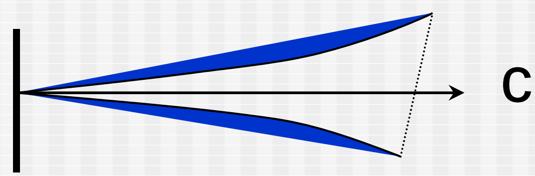
$$[\Phi]=[\Psi]\left[m_r^{-1/2}\right]=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\left[m_r^{-1/2}\right]=\begin{bmatrix}.707 & .707\\ .707 & -.707\end{bmatrix}$$

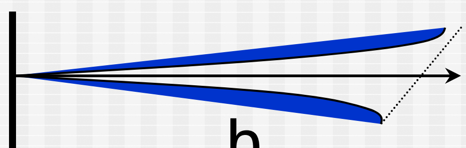
Multiple modes

- The situation where two (or more) modes have the same natural frequency.
- It occurs in structures with a degree of symmetry, such as discs, rings, cylinders.
- Free vibration at such frequency may occur not only in each of the two modes but also in a linear combination of them.









a Vertical mode

b Horizontal mode

c Oblique mode

Forced Response of MDOF

$$[M]\{\ddot{x}\}+[K]\{x\}=\{f(t)\}$$

$$([K]-\omega^2[M])\{X\}e^{i\omega t}=\{F\}e^{i\omega t}$$

Or by rearranging:

$$\{X\}=([K]-\omega^2[M])^{-1}\{F\}$$

Which may be written as:

$$\{X\}=[H(\omega)]\{F\}$$ **Response model**

$$H_{jk}(\omega)=\frac{X_j}{F_k} \quad F_m=0 \quad for\ all \quad m \quad except\ m=k$$

Forced Response of MDOF

The values of matrix [H] can be computed easily at each frequency point. However, this has several advantages:

- It becomes costly for large N.
- It is inefficient if only a few FRF expression is required.
- It provides no insight into the form of various FRF properties.

Therefore, we make use of modal properties for deriving the FRF parameters instead of spatial properties.

Forced Response of MDOF

$$([K]-\omega^2[M])=[H(\omega)]^{-1}$$

Premultiply both sides by $\left[\Phi\right]^T$ and postmultiply by $\left[\Phi\right]$

$$\left[\Phi\right]^T([K]-\omega^2[M])\left[\Phi\right]=\left[\Phi\right]^T[H(\omega)]^{-1}\left[\Phi\right]$$

$$\left[\omega_r^2-\omega^2\right]=\left[\Phi\right]^T[H(\omega)]^{-1}\left[\Phi\right]$$

$$[H(\omega)]=\left[\Phi\right]\underbrace{\left[\omega_r^2-\omega^2\right]}^{-1}\left[\Phi\right]^T$$

Equation 1

Note that:

Diagonal matrix

$$[\Phi]^T[\Phi]=[\Phi][\Phi]^T=[I]$$

Forced Response of MDOF

As H is a symmetric matrix then:

$$H_{jk} = H_{kj}$$

or

$$X_j / F_k = X_k / F_j$$

Principle of reciprocity

Using equation 1:

$$H_{jk} = \sum_{r=1}^{N} \frac{\phi_{jr}\phi_{kr}}{\omega_r^2 - \omega^2} = \sum_{r=1}^{N} \frac{\psi_{jr}\psi_{kr}}{m_r(\omega_r^2 - \omega^2)}$$

or

$$H_{jk} = \sum_{r=1}^{N} \frac{{}_rA_{jk}}{\omega_r^2 - \omega^2}$$

Modal constant

Forced Response of 2DOF

$$[-m_1\omega^2+(k_1+k_2)]X_1-k_2X_2=F_1$$

$$[-m_2\omega^2+(k_2+k_3)]X_2-k_2X_1=F_2$$

Which gives:

$$\left(\frac{X_1}{F_1}\right)_{F_2=0}=\frac{k_2+k_3-m_2\omega^2}{m_1m_2\omega^4-((m_1+m_2)k_2+m_1k_3+m_2k_1)\omega^2+(k_1k_2+k_1k_3+k_2k_3)}$$

Numerically:

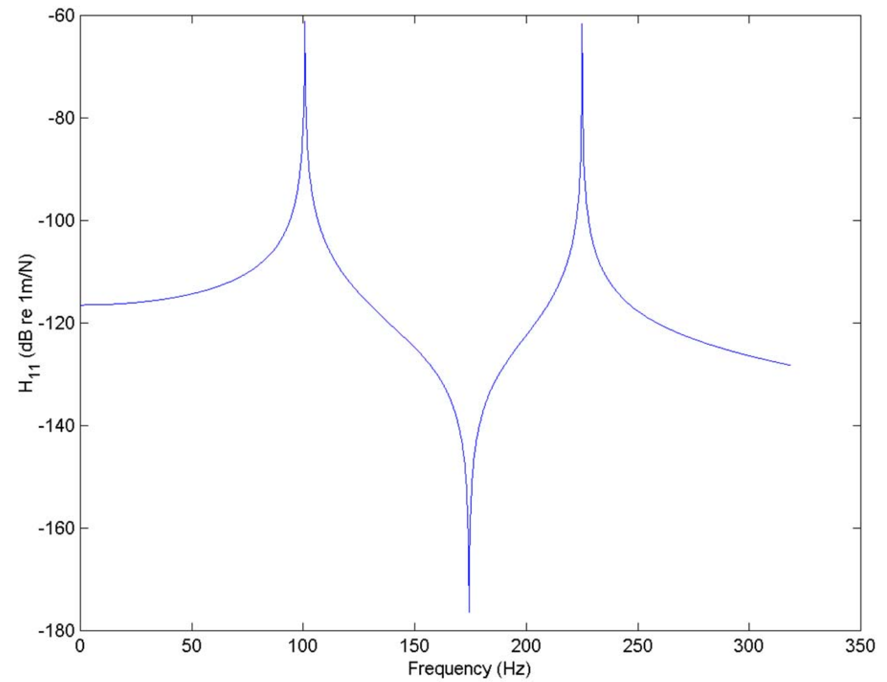
$$\left(\frac{X_1}{F_1}\right)_{F_2=0}=\frac{1.2\times10^6-\omega^2}{\omega^4-2.4\times10^6\omega^2+0.8\times10^{12}}$$

Forced Response of 2DOF

$$H_{11} = \left(\frac{X_1}{F_1}\right) = \frac{(\phi_{11})^2}{\omega_1^2 - \omega^2} + \frac{(\phi_{12})^2}{\omega_2^2 - \omega^2}$$

Or numerically:

$$H_{11} = \frac{0.5}{0.4\times10^6 - \omega^2} + \frac{0.5}{2\times10^6 - \omega^2}$$



Receptance FRF (H_{11}) for 2dof system