MDOF Systems with Proportional Damping

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General Concepts

- Proportional damping has the advantage of being easy to include in the analysis so far.
- The modes of a structure with proportional damping are almost identical to those of the undamped version of the model.
- It is possible to derive the modal properties of a proportionally damped system by analyzing the undamped version in full and then making a correction for the damping.

Proportional damping – special case

Consider the general equation of motion for a MDOF with a viscous damping:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{f\}$$

Assume that the damping matrix is directly proportional to the Stiffness matrix:

$$[C]=\beta[K]$$

Then:

$$[\Psi]^T[C][\Psi]=\beta[\Psi]^T[K][\Psi]=\beta[k_r]=[c_r]$$

Proportional damping – special case

Let's define {p} as:

The undamped modal matrix

$$\{x\} = [\Psi]\{p\}$$

Then the equation of motion becomes (f=0):

$$[m_r]\{\ddot{p}\} + [c_r]\{\dot{p}\} + [k_r]\{p\} = \{0\}$$

From which:

$$m_r \ddot{p}_r + c_r \dot{p}_r + k_r p_r = 0$$

This is a SDOF system and therefore:

$$\omega_r^2 = \frac{k_r}{m_r}, \quad \omega_d = \omega_r \sqrt{1-\zeta_r^2}\,, \qquad \zeta_r = \frac{c_r}{2\sqrt{k_r m_r}} = 0.5\beta\omega_r$$

Proportional damping – special case

The receptane matrix can be defined as:

$$[H(\omega)] = ([K] + i\omega[C] - \omega^2[M])^{-1}$$

or

$$H_{jk}(\omega) = \sum_{r=1}^{N} \frac{\psi_{jr}\psi_{kr}}{(k_r - \omega^2 m_r) + i\omega c_r}$$

Proportional damping – general case

The general form of proportional damping is:

$$[C] = \beta[K] + \gamma[M]$$

Again assume:

$$\{x\} = [\Psi]\{p\}$$

And then:

$$[\Psi]^T[C][\Psi] = \beta[k_r] + \gamma[m_r]$$

In this case, the damped system has eigenvalues and eigenvectors as follow:

$$\omega_d = \omega_r\sqrt{1-\zeta_r^2}\ , \qquad \zeta_r = 0.5\beta\omega_r + 0.5\gamma/\omega_r$$

Proportional Hysteretic damping

Consider the general equation of motion for a MDOF with a hysteretic damping:

$$[M]\{\ddot{x}\}+([K]+i[D])\{x\}=\{f\}$$

Assume the hysteretic damping as:

$$[D]=\beta[K]+\gamma[M]$$

And again

$$\omega_r^2=\frac{k_r}{m_r}, \quad \lambda_r^2=\omega_r^2(1+i\eta_r), \quad \eta_r=\beta+\gamma/\omega_r^2$$

Exercise: Extract the above relations from the equation of motion.